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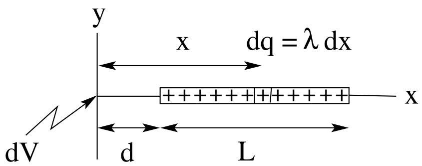
06. Electric potential and electric field

Gerhard Müller
University of Rhode Island, gmuller@uri.edu

Robert Coyne
University of Rhode Island, robcoyne@uri.edu

Electric Potential of Charged Rod

- Charge per unit length: $\lambda = Q/L$
- Charge on slice dx: $dq = \lambda dx$



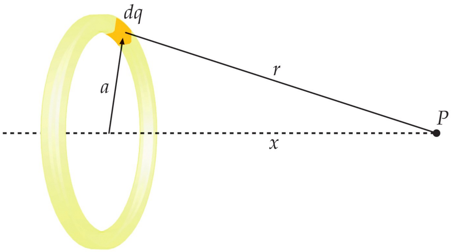
- Electric potential generated by slice dx: $dV = \dfrac{kdq}{x} = \dfrac{k\lambda dx}{x}$
- Electric potential generated by charged rod:

$$V = k\lambda \int_d^{d+L} \frac{dx}{x} = k\lambda \Big[\ln x \Big]_d^{d+L} = k\lambda \left[\ln(d+L) - \ln d\right] = k\lambda \ln \frac{d+L}{d}$$

- Limiting case of very short rod $(L \ll d)$: $V = k\lambda \ln\left(1 + \dfrac{L}{d}\right) \simeq k\lambda \dfrac{L}{d} = \dfrac{kQ}{d}$

Electric Potential of Charged Ring

- Total charge on ring: Q
- Charge per unit length: $\lambda = Q/2\pi a$
- Charge on arc: dq

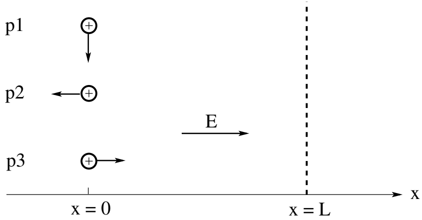


Find the electric potential at point P on the axis of the ring.

- $dV = k\,\dfrac{dq}{r} = \dfrac{kdq}{\sqrt{x^2 + a^2}}$
- $V(x) = k\displaystyle\int \dfrac{dq}{\sqrt{x^2 + a^2}} = \dfrac{k}{\sqrt{x^2 + a^2}}\displaystyle\int dq = \dfrac{kQ}{\sqrt{x^2 + a^2}}$

Electric Potential and Potential Energy: Application (6)

Three protons are projected from $x = 0$ with equal initial speed v_0 in different directions. They all experience the force of a uniform horizontal electric field $\vec{E}$. Ultimately, they all hit the vertical screen at $x = L$. Ignore gravity.



(a) Which proton travels the longest time?

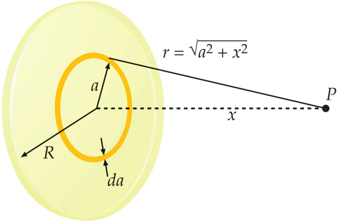
(b) Which proton travels the longest path?

(c) Which particle has the highest speed when it hits the screen?

Two of the questions are easy, one is hard.

Electric Potential of Charged Disk

- Area of ring: $2\pi a da$
- Charge on ring: $dq = \sigma(2\pi a da)$
- Charge on disk: $Q = \sigma(\pi R^2)$



Find the electric potential at point P on the axis of the disk.

- $dV = k\,\dfrac{dq}{\sqrt{x^2+a^2}} = 2\pi\sigma k\,\dfrac{a da}{\sqrt{x^2+a^2}}$
- $V(x) = 2\pi\sigma k \displaystyle\int_0^R \dfrac{a da}{\sqrt{x^2+a^2}} = 2\pi\sigma k\left[\sqrt{x^2+a^2}\right]_0^R = 2\pi\sigma k\left[\sqrt{x^2+R^2} - |x|\right]$

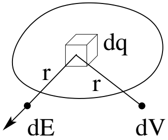
Electric potential at large distances from the disk $(|x| \gg R)$:

$$V(x) = 2\pi\sigma k|x|\left[\sqrt{1+\frac{R^2}{x^2}} - 1\right] \simeq 2\pi\sigma k|x|\left[1 + \frac{R^2}{2x^2} - 1\right] = \frac{k\sigma\pi R^2}{|x|} = \frac{kQ}{|x|}$$

Electric Field and Electric Potential

Determine the field or the potential from the source (charge distribution):

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r^2}\hat{r}$$



$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r}$$

Determine the field from the potential: $\vec{E} = -\frac{\partial V}{\partial x}\hat{\imath} - \frac{\partial V}{\partial y}\hat{\jmath} - \frac{\partial V}{\partial z}\hat{k}$

Determine the potential from the field: $V = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{s}$

- Systems with $\vec{E} = E_x(x)\hat{\imath}$: $E_x = -\frac{dV}{dx} \quad \Leftrightarrow \quad V(x) = -\int_{x_0}^{x} E_x dx$
- Application to charged ring: $E_x = \frac{kQx}{(x^2+a^2)^{3/2}} \Leftrightarrow V = \frac{kQ}{\sqrt{x^2+a^2}}$
- Application to charged disk (at $x > 0$): $E_x = 2\pi\sigma k\left[1 - \frac{x}{\sqrt{x^2+R^2}}\right] \Leftrightarrow V = 2\pi\sigma k\left[\sqrt{x^2+R^2} - x\right]$

For given electric potential $V(x)$find the electric field

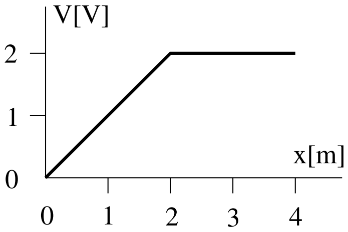
(a) $E_x(1\text{m})$,

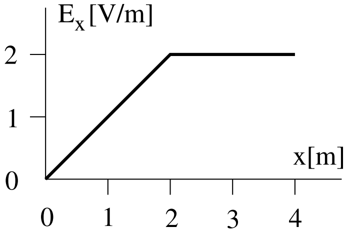
(b) $E_x(3\text{m})$.

For given electric field $E_x(x)$ and given reference potential potential $V(0) = 0$ find the electric potential

(c) $V(2\text{m})$,

(d) $V(4\text{m})$.





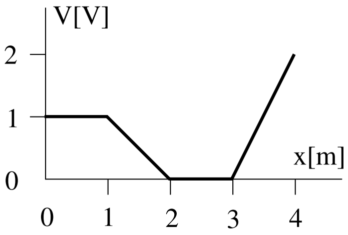
For given electric potential $V(x)$ find the electric field

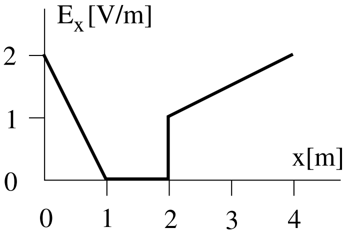
(a) $E_x(0.5\text{m})$, (b) $E_x(1.5\text{m})$,

(c) $E_x(2.5\text{m})$, (d) $E_x(3.5\text{m})$.

For given electric field $E_x(x)$ and given reference potential potential $V(0) = 0$ find the electric potential

(e) $V(1\text{m})$, (f) $V(2\text{m})$, (g) $V(4\text{m})$.

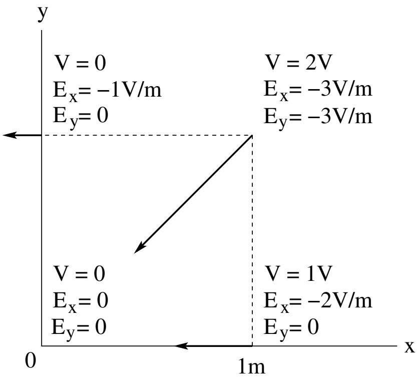




Electric Field from Electric Potential in Two Dimensions

- Given is the electric potential: $V(x,y) = ax^2 + bxy^3$ with $a = 1\mathrm{V/m}^2$, $b = 1\mathrm{V/m}^4$.
- Find the electric field: $\vec{E}(x,y) = E_x(x,y)\hat{\imath} + E_y(x,y)\hat{\jmath}$ via partial derivatives.

$$E_x = -\frac{\partial V}{\partial x} = -2ax - by^3, \qquad E_y = -\frac{\partial V}{\partial y} = -3bxy^2$$



Electric Potential from Electric Field in Two Dimensions

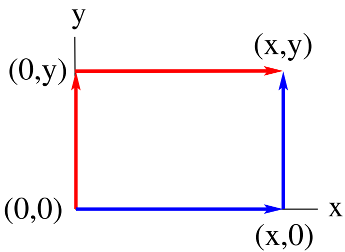
- Given is the electric field: $\vec{E} = -(2ax + by^3)\hat{\imath} - 3bxy^2\hat{\jmath}$ with $a = 1\mathrm{V/m^2}$, $b = 1\mathrm{V/m^4}$.
- Find the electric potential $V(x,y)$ via integral along a specific path:

Red path $(0,0) \rightarrow (0,y) \rightarrow (x,y)$:

$$\begin{aligned} V(x,y) &= -\int_0^y E_y(0,y)dy - \int_0^x E_x(x,y)dx \\ &= 0 + \int_0^x (2ax + by^3)dx = ax^2 + bxy^3 \end{aligned}$$

Blue path $(0,0) \rightarrow (x,0) \rightarrow (x,y)$:

$$\begin{aligned} V(x,y) &= -\int_0^x E_x(x,0)dx - \int_0^y E_y(x,y)dy \\ &= \int_0^x (2ax)dx + \int_0^y (3bxy^2)dy = ax^2 + bxy^3 \end{aligned}$$

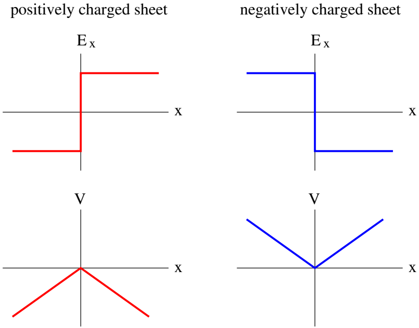


Electric Potential of a Charged Plane Sheet

Consider an infinite plane sheet perpendicular to the x-axis at $x = 0$.

The sheet is uniformly charged with charge per unit area σ.

- Electric field (magnitude): $E = 2\pi k|\sigma| = \dfrac{|\sigma|}{2\epsilon_0}$
- Direction: away from (toward) the sheet if $\sigma > 0$ ($\sigma < 0$).
- Electric field (x-component):
  $E_x = \pm 2\pi k\sigma$.
- Electric potential:
  $V = -\displaystyle\int_0^x E_x dx = \mp 2\pi k\sigma x.$
- Here we have used $x_0 = 0$
  as the reference coordinate.



Electric Potential of a Uniformly Charged Spherical Shell

- Electric charge on shell: $Q = \sigma A = 4\pi\sigma R^2$

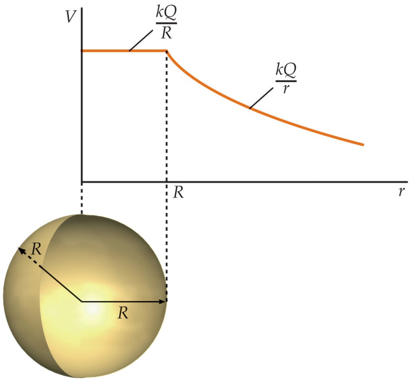
- Electric field at $r > R$: $E = \dfrac{kQ}{r^2}$
- Electric field at $r < R$: $E = 0$
- Electric potential at $r > R$:

$$V = -\int_{\infty}^{r} \frac{kQ}{r^2}\,dr = \frac{kQ}{r}$$

- Electric potential at $r < R$:

$$V = -\int_{\infty}^{R} \frac{kQ}{r^2}\,dr - \int_{R}^{r} (0)dr = \frac{kQ}{R}$$

- Here we have used $r_0 = \infty$ as the reference value of the radial coordinate.



Electric Potential of a Uniformly Charged Solid Sphere

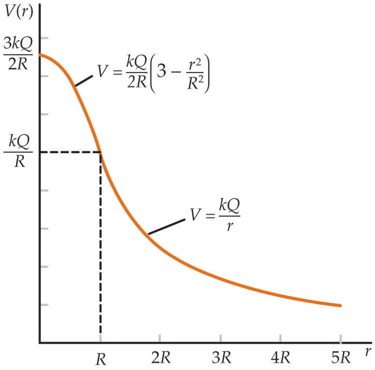
- Electric charge on sphere: $Q = \rho V = \dfrac{4\pi}{3}\rho R^3$
- Electric field at $r > R$: $E = \dfrac{kQ}{r^2}$
- Electric field at $r < R$: $E = \dfrac{kQ}{R^3}\, r$
- Electric potential at $r > R$:

$$V = -\int_\infty^r \frac{kQ}{r^2}\, dr = \frac{kQ}{r}$$

- Electric potential at $r < R$:

$$V = -\int_\infty^R \frac{kQ}{r^2}\, dr - \int_R^r \frac{kQ}{R^3}\, r\, dr$$

$$\Rightarrow V = \frac{kQ}{R} - \frac{kQ}{2R^3}\left(r^2 - R^2\right) = \frac{kQ}{2R}\left(3 - \frac{r^2}{R^2}\right)$$



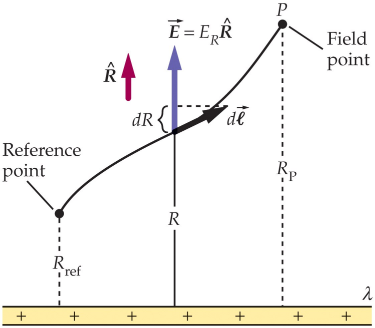
Electric Potential of a Uniformly Charged Wire

- Consider a uniformly charged wire of infinite length.
- Charge per unit length on wire: λ (here assumed positive).
- Electric field at radius r: $E = \dfrac{2k\lambda}{r}$.
- Electric potential at radius r:

$$V = -2k\lambda \int_{r_0}^{r} \frac{1}{r}\, dr = -2k\lambda \left[\ln r - \ln r_0\right]$$

$$\Rightarrow V = 2k\lambda \ln \frac{r_0}{r}$$

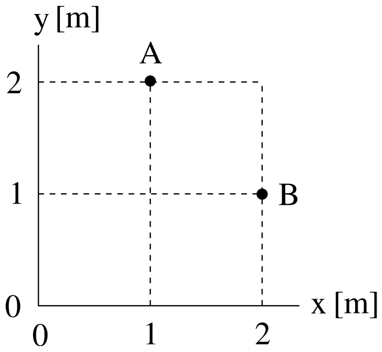
- Here we have used a finite, nonzero reference radius $r_0 \neq 0, \infty$.
- The illustration from the textbook uses R_{ref} for the reference radius, R for the integration variable, and R_p for the radial position of the field point.



Given is the electric potential $V(x,y) = cxy^2$ with $c = 1\text{V/m}^3$.

(a) Find the value (in SI units) of the electric potential V at point A.

(b) Find the components E_x, E_y (in SI units) of the electric field at point B.



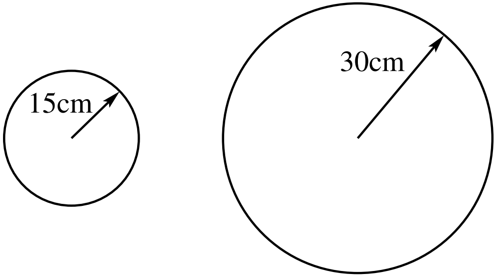
Electric Potential of Conducting Spheres (2)

Consider a conducting sphere with radius $r = 15\text{cm}$ and electric potential $V = 200\text{V}$ relative to a point at infinity.

(a) Find the charge Q and the surface charge density σ on the sphere.

(b) Find the magnitude of the electric field E just outside the sphere.

(c) What happens to the values of Q, V, σ, E when the radius of the sphere is doubled?

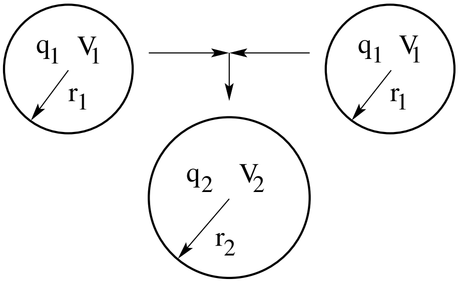


Electric Potential of Conducting Spheres (3)

A spherical raindrop of 1mm diameter carries a charge of 30pC.

(a) Find the electric potential of the drop relative to a point at infinity under the assumption that it is a conductor.

(b) If two such drops of the same charge and diameter combine to form a single spherical drop, what is its electric potential?



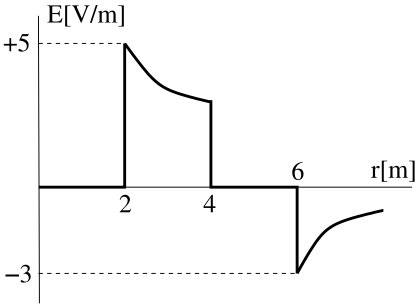
Electric Potential of Conducting Spheres (1)

A conducting sphere of radius $r_1 = 2$m is surrounded by a concentric conducting spherical shell of radii $r_2 = 4$m and $r_3 = 6$m. The graph shows the electric field $E(r)$.

(a) Find the charges q_1, q_2, q_3 on the three conducting surfaces.

(b) Find the values V_1, V_2, V_3 of the electric potential on the three conducting surfaces relative to a point at infinity.

(c) Sketch the potential $V(r)$.



Electric Potential of Conducting Spheres (4)

A positive charge is distributed over two conducting spheres 1 and 2 of unequal size and connected by a long thin wire. The system is at equilibrium.

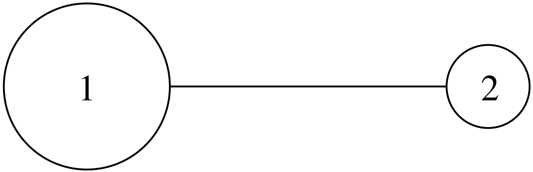
Which sphere (1 or 2)...

(a) carries more charge on its surface?

(b) has the higher surface charge density?

(c) is at a higher electric potential?

(d) has the stronger electric field next to it?



Electric Potential Energy of Two Point Charges

Consider two different perspectives:

#1a Electric potential when q_1 is placed: $V(\vec{r}_2) \doteq V_2 = k\dfrac{q_1}{r_{12}}$

Electric potential energy when q_2 is placed into potential V_2: $U = q_2 V_2 = k\dfrac{q_1 q_2}{r_{12}}$

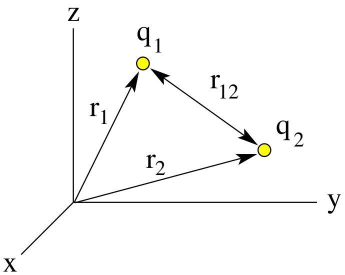
#1b Electric potential when q_2 is placed: $V(\vec{r}_1) \doteq V_1 = k\dfrac{q_2}{r_{12}}$

Electric potential energy when q_1 is placed into potential V_1: $U = q_1 V_1 = k\dfrac{q_1 q_2}{r_{12}}$.

#2 Electric potential energy of q_1 and q_2:

$$U = \frac{1}{2}\sum_{i=1}^{2} q_i V_i,$$

where $V_1 = k\dfrac{q_2}{r_{12}}, \quad V_2 = k\dfrac{q_1}{r_{12}}.$



Electric Potential Energy of Three Point Charges

#1 Place q_1, then q_2, then q_3, and add all changes in potential energy:

$$U = 0 + k\frac{q_1q_2}{r_{12}} + k\left(\frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right) = k\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right).$$

#2 Symmetric expression of potential energy U in terms of the potentials V_i experienced by point charges q_1:

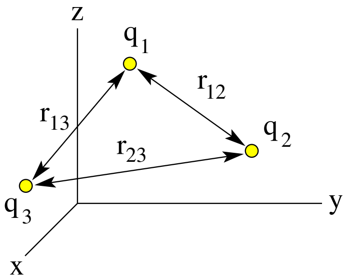
$$U = \frac{1}{2}\sum_{i=1}^{3} q_iV_i = k\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right),$$

where

$$V_1 = k\left(\frac{q_2}{r_{12}} + \frac{q_3}{r_{13}}\right),$$

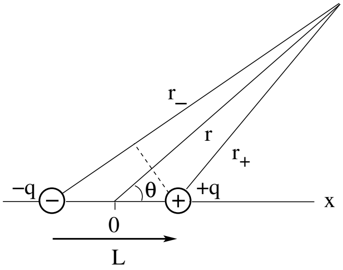
$$V_2 = k\left(\frac{q_1}{r_{12}} + \frac{q_3}{r_{23}}\right),$$

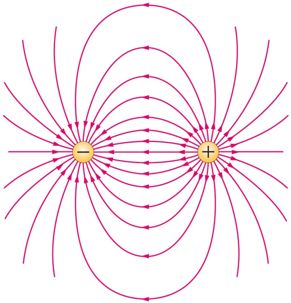
$$V_3 = k\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right).$$



Electric Dipole Potential

- Use spherical coordinates: $V = V(r,\theta)$ independent of azimuthal coordinate ϕ.
- Superposition principle: $V = V_+ + V_- = k\left(\frac{q}{r_+} + \frac{(-q)}{r_-}\right) = kq\,\frac{r_- - r_+}{r_- r_+}$
- Large distances $(r \gg L)$: $r_- - r_+ \simeq L\cos\theta,\ r_- r_+ \simeq r^2 \Rightarrow\ V(r,\theta) \simeq k\,\frac{qL\cos\theta}{r^2}$
- Electric dipole moment: $p = qL \quad$ (magnitude)
- Electric dipole potential: $V(r,\theta) \simeq k\,\frac{p\cos\theta}{r^2}$





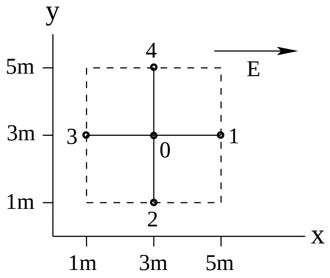
Unit Exam I: Problem #3 (Spring ’11)

Consider a region of space with a uniform electric field $\mathbf{E} = 0.5\mathrm{V/m}\,\hat{\mathbf{i}}$. Ignore gravity.

(a) If the electric potential vanishes at point 0, what are the electric potentials at points 1 and 2?

(b) If an electron ($m = 9.11 \times 10^{-31}$kg, $q = -1.60 \times 10^{-19}$C) is released from rest at point 0, toward which point will it start moving?

(c) What will be the speed of the electron when it gets there?



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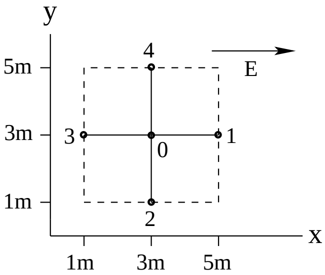

Solution:

(a) $V_1 = -(0.5\mathrm{V/m})(2\mathrm{m}) = -1\mathrm{V}, \quad V_2 = 0.$

Consider a region of space with a uniform electric field $\mathbf{E} = 0.5\text{V/m}\,\hat{\mathbf{i}}$. Ignore gravity.

(a) If the electric potential vanishes at point 0, what are the electric potentials at points 1 and 2?

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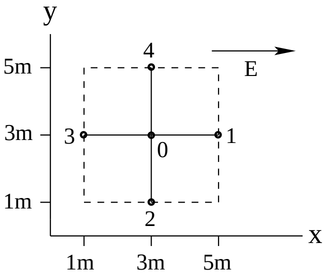

Solution:

(a) $V_1 = -(0.5\text{V/m})(2\text{m}) = -1\text{V}, \quad V_2 = 0.$

(b) $\mathbf{F} = q\mathbf{E} = -|qE|\hat{\mathbf{i}}$ (toward point 3).

Consider a region of space with a uniform electric field $\mathbf{E} = 0.5\mathrm{V/m}\,\hat{\mathbf{i}}$. Ignore gravity.

(a) If the electric potential vanishes at point 0, what are the electric potentials at points 1 and 2?

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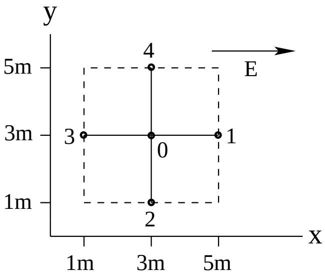


Solution:

(a) $V_1 = -(0.5\mathrm{V/m})(2\mathrm{m}) = -1\mathrm{V}, \quad V_2 = 0.$

(b) $\mathbf{F} = q\mathbf{E} = -|qE|\hat{\mathbf{i}}$ (toward point 3).

(c) $\Delta V = (V_3 - V_0) = 1\mathrm{V}, \quad \Delta U = q\Delta V = -1.60 \times 10^{-19}\mathrm{J},$

$K = -\Delta U = 1.60 \times 10^{-19}\mathrm{J}, \quad v = \sqrt{\dfrac{2K}{m}} = 5.93 \times 10^5\mathrm{m/s}.$

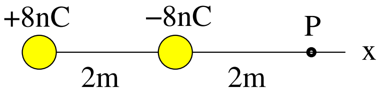
Alternatively:

$F = qE = 8.00 \times 10^{-20}\mathrm{N}, \quad a = \dfrac{F}{m} = 8.78 \times 10^{10}\mathrm{m/s^2},$

$|\Delta x| = 2\mathrm{m}, \quad v = \sqrt{2a|\Delta x|} = 5.93 \times 10^5\mathrm{m/s}.$

Consider two point charges positioned on the x-axis as shown.

(a) Find magnitude and direction of the electric field at point P.
(b) Find the electric potential at point P.
(c) Find the electric potential energy of an electron (mass $m = 9.1 \times 10^{-31}$kg, charge $q = -1.6 \times 10^{-19}$C) when placed at point P.
(d) Find magnitude and direction of the acceleration the electron experiences when released at point P.



Consider two point charges positioned on the x-axis as shown.

(a) Find magnitude and direction of the electric field at point P.
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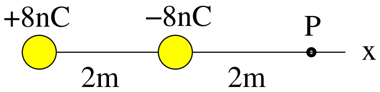

Solution:

(a) $E_x = +k\frac{8\text{nC}}{(4\text{m})^2} + k\frac{(-8\text{nC})}{(2\text{m})^2} = 4.5\text{N/C} - 18\text{N/C} = -13.5\text{N/C}$ (directed left).

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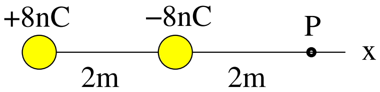

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(b) $V = +k\frac{8\text{nC}}{4\text{m}} + k\frac{(-8\text{nC})}{2\text{m}} = 18\text{V} - 36\text{V} = -18\text{V}.$

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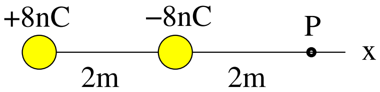

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(c) $U = qV = (-18\text{V})(-1.6 \times 10^{-19}\text{C}) = 2.9 \times 10^{-18}\text{J}.$

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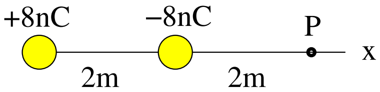

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(a) $E_x = +k\dfrac{8\text{nC}}{(4\text{m})^2} + k\dfrac{(-8\text{nC})}{(2\text{m})^2} = 4.5\text{N/C} - 18\text{N/C} = -13.5\text{N/C}$ (directed left).

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(c) $U = qV = (-18\text{V})(-1.6 \times 10^{-19}\text{C}) = 2.9 \times 10^{-18}\text{J}.$

(d) $a_x = \dfrac{qE_x}{m} = \dfrac{(-1.6 \times 10^{-19}\text{C})(-13.5\text{N/C})}{9.1 \times 10^{-31}\text{kg}} = 2.4 \times 10^{12}\text{ms}^{-2}$ (directed right).

Electric Potential and Potential Energy: Application (9)

Consider four point charges of equal magnitude positioned at the corners of a square as shown. Answer the following questions for points A, B, C.

(1) Which point is at the highest electric potential?

(2) Which point is at the lowest electric potential?

(3) At which point is the electric field the strongest?

(4) At which point is the electric field the weakest?